Neutrino Oscillations and Associated Symmetries

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Neutrinos are near-massless fermions that interact via the weak nuclear force only. Experiments at neutrino observatories such as the Super-Kamiokande have shown that neutrinos can undergo flavour transformations as they propagate through space. In this work, we aim to study the reasons and derive the probabilities of such oscillations for both the two- and three- neutrino cases and analyse the results to make a few observations about the physical properties of neutrinos. Furthermore, we also look at the symmetries that are associated with this phenomenon, particularly that of CPT invariance and CP violation by quantifying the asymmetry in terms of the Jarlskog invariant.

Keywords: Quantum Mechanics, Particle Physics, Neutrino Oscillation, CP violation

I. INTRODUCTION

The Standard Model is a theory that describes three out of the four fundamental forces in the Universe and classifies elementary particles into distinct groups. Generally, these particles can be distinctly classified into two major classes: bosons — particles that obey Bose-Einstein statistics and have an integer spin; and fermions — particles that obey Fermi-Dirac statistics and have half-integer spin. Another form of classification can be made on the basis of the interactions that a particle undergoes. Particles that undergo strong interactions (with the strong nuclear force) are called hadrons and the ones that do not undergo any such interactions are called leptons. The particle we are interested in is the neutrino, which is a lepton. The neutrino is an electrically-neutral fermion that only interacts via the weak nuclear force and gravity. As the weak nuclear force has a very short range and gravitational interactions are extremely weak at the subatomic level, neutrinos are usually elusive and hard to detect.

Neutrinos can occur in one of three different types or "flavours" (although a more accurate term is generation): electron neutrinos (ν_e), muon neutrinos (ν_μ), or tau neutrinos (ν_τ); each corresponding to the fermions involved in their creation or annihilation [1]. These forms are usually observed when a neutrino interacts with other particles. For every neutrino, there also exists a corresponding antiparticle called the *antineutrino* (denoted by $\bar{\nu}$) which is also an electrically-neutral fermion, although with an opposite chirality (we briefly cover chirality in the section on symmetries).

Neutrinos also have some peculiar properties, with once such property being — as a well-known science educator put it — that neutrinos are "identity agnostic" [2]. When neutrinos travel through space, they can be observed in one of three mass states. When they interact with matter, however, they are observed to be in their flavour states. Interestingly, there is no one-to-one correspondence between these two "identities". It is possible for a neutrino which was produced in a certain flavour to later be measured in a completely different flavour. For example, if an electron neutrino was created in a decay reaction, then it is possible for the same neutrino to be measured as a tau neutrino at a later point in space. This phenomenon — called neutrino oscillation — was first predicted by Italian-Soviet physicist Bruno Pontecorvo in 1957 [3, 4] and experimentally verified in 1998 and 2001 by Takaaki Kajita of the Super-Kamiokande Observatory and Arthur McDonald of Sudbury Neutrino Observatory for which they received the 2015 Nobel Prize in Physics [5].

In this work, we study the reasons behind neutrino oscillations and derive the probabilities of neutrino flavour changes in vacuum for both the two and three neutrino cases. We also mention the parameters affecting this probability followed by a discussion of some basic properties of neutrinos which can be inferred from the derivations made. Finally, we discuss this problem from the point of view of symmetries and look at the symmetries that are associated with this problem. We also quantify the asymmetries that occur and briefly state the properties that can be derived from the violation of such symmetries. We will be using natural units ($\hbar = c = 1$) throughout this paper unless specified otherwise.

II. LEPTONIC MIXING

As we saw earlier, neutrinos can be found in three distinct flavours: electron, muon and tau neutrinos. Each flavour has a corresponding antineutrino associated with it. We also mentioned that neutrinos are capable of flavour change. As it turns out, this oscillatory behaviour happens because neutrino flavours are actually a superposition of different neutrino masses. This is called *leptonic mixing* because the neutrino flavours are a mixture of neutrino masses and vice-versa. Let us define this more formally. Consider that there is a spectrum of neutrino mass eigenstates $\{\nu_i\}, i = 1, 2, 3$ each corresponding to an eigenvalue m_i which is the

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mass of the neutrino. The flavours of the neutrino form another orthogonal basis like the mass does. These can be denoted by $\{\nu_\alpha\}, \alpha = e, \mu, \tau$. The ability to express the three flavours of the neutrino as a superposition of mass eigenstates is called mixing. For example, consider a leptonic decay of the form $W^+ \to \nu_i + \bar{\ell}_\alpha$ where ℓ_α is a charged lepton of a flavour α. Due to mixing, the neutrino that is formed in the eigenstate ν_i is not necessarily of the same flavour α but is in fact a superposition of multiple flavours — out of which one is observed upon measurement. Mathematically, the consequence of mixing is that the mass matrix is not diagonal when written in the flavour basis.

There must exist a basis transformation matrix between the mass basis and flavour basis. Let us denote this by U. This unitary transformation which relates the flavour eigenstates to the mass eigenstates is called the *lepton mixing matrix*, or the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [6]. Therefore,

$$\begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \\ \nu_\tau(x) \end{pmatrix}_L = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}_L \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \\ \nu_3(x) \end{pmatrix}_L \tag{1}$$

The subscript L is thrown in for some pedanticism. It denotes that we are dealing with *left-handed* neutrinos. We will briefly mention chirality in the section on symmetry. Another minor detail to note at this point is that the quantities ν_α or ν_i are actually neutrino *fields* with an explicit position dependence; however such details are out of the scope of this paper. We drop the L and x for the sake of notational simplicity.
Eq. (1) can be written more compactly as

$$\nu_\alpha = \sum_i U_{\alpha i} \nu_i \tag{2}$$

where $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$.

Naturally, since the PMNS matrix is a basis transformation matrix, it is unitary. Therefore $UU^\dagger = U^\dagger U = \mathbb{1}$

$$\begin{aligned} \sum_i U_{\alpha i} U^*_{\beta i} &= \delta_{\alpha\beta} \quad (\alpha, \beta = e, \mu, \tau) \\ \sum_\alpha U_{\alpha i} U^*_{\alpha j} &= \delta_{ij} \quad (i, j = 1, 2, 3) \end{aligned} \tag{3}$$

The PMNS matrix is usually parameterised in terms of three angles θ_{12}, θ_{23}, and θ_{13} called the mixing angles; and a phase δ_{CP} called the CP violation phase whose relevance would be clear in later sections [7, 8]. The PMNS matrix can then be written as

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{-i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{-i\delta_{CP}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{-i\delta_{CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{-i\delta_{CP}} & c_{13}c_{23} \end{pmatrix} \tag{4}$$

where $c_{ij} \equiv \cos(\theta_{ij})$ and $s_{ij} \equiv \sin(\theta_{ij})$. Without loss of generality, $\theta_{ij} \in \left[0, \frac{\pi}{2}\right]$ and $\delta_{CP} \in [0, 2\pi)$

It is evident from Eq. (2) that the mass-i fraction in flavour α and the flavour α fraction in mass-i is $|U_{\alpha i}|^2$. Therefore, the probability of measuring a mass i when a lepton ℓ_α is produced in the leptonic decay mentioned above is $|U_{\alpha i}|^2$.

III. NEUTRINO OSCILLATIONS IN VACUUM

Neutrinos interact very weakly with matter which makes it very difficult to detect them; however, a charged lepton (like an electron) that is produced alongside a neutrino can easily be detected and its flavour can be identified. Therefore, we can also determine the flavour of the neutrino, which is say α. In a similar fashion, the neutrino can travel a path length L and then interact with a detector to produce another charged lepton. This time, say the flavour identified was β. If it is found that $\alpha \neq \beta$, then a flavour change has occurred. This change, $\alpha \to \beta$ is a quantum mechanical phenomenon and thus, we can find its probability, $P(\nu_\alpha \to \nu_\beta)$.

III.1. Finding the Oscillation Probability

Usually, an ideal oscillation experiment would involve three steps as elucidated before. The first is a production of a pure flavour state from a decay process, for example a charged pion decay that produces ν_μ, $\quad \pi^+ \to \nu_\mu + \mu^+$. This flavour eigenstate is a superposition of the mass eigenstates and the coefficients are given by the PMNS matrix [1]:

[1] This is where that distinction between neutrinos and neutrino fields from Eq. (2) comes into the picture. The relation between the flavour and mass eigenstates for neutrino *states* involves the complex conjugate of the PMNS matrix, as opposed to the matrix itself for neutrino *fields*.

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i} |\nu_i\rangle \tag{5}$$

The second step is the propagation of the neutrino. Each mass eigenstate is an eigenstate of the Hamiltonian. In its rest frame, the time evolution of the neutrino state is given by the Schrödinger equation

$$i\frac{\partial}{\partial t} |\nu_i\rangle = E_i |v_i\rangle \tag{6}$$

where E_i is the energy of the m_ith mass. Therefore,

$$|\nu_i(t)\rangle = |\nu_i(0)\rangle e^{-iE_i t} \tag{7}$$

Therefore, each mass eigenstate evolves with its own phase factor. Since neutrinos for all practical purposes are hyper-relativistic, that is, $v \approx c$, one can use the relation $E_i = \sqrt{p_i^2 + m_i^2}$, where p_i is the momentum of the i-th mass eigenstate, to make the following approximation:

$$E_i = \sqrt{p_i^2 + m_i^2} \approx p_i + \frac{m_i^2}{2p_i} \approx E + \frac{m_i^2}{2E} \tag{8}$$

It should be noted that we have used the same value of energy for all mass eigenstates. This can be justified by the following argument: assume that two different components ν_i and ν_j have energies E_i and E_j. When they reach the detector, they have phases $e^{-iE_i t}$ and $e^{-iE_j t}$ respectively. Since the detector can only measure relative phases, it measures the quantity $e^{-i(E_i - E_j)t}$, which over time averages to zero. Therefore, only components with equal energies are detected [9].

If the neutrino travels a distance L, then the time taken for the propagation is $t \approx L/c = L$. Therefore, the probability amplitude for propagation is $\text{Prop}(\nu_i) = \exp\left(-i\frac{m_i^2}{2E}L\right)$.

The final step is the detection. Assume that the neutrino interacts with the detector to produce a charged lepton $\bar{\ell}_\beta$. Therefore, the probability amplitude for it to exist in the mass eigenstate ν_i is $U_{\beta i}$. Therefore, this entire process modifies the coherent superposition that was produced initially and at a later time t, the neutrino is no longer in a pure flavour eigenstate but a superposition given by

$$|\nu(t)\rangle = \sum_i U^*_{\alpha i} \exp\left(-i\frac{m_i^2}{2E}L\right) U_{\beta i} |\nu_i\rangle \tag{9}$$

The probability amplitude for a flavour α neutrino to have oscillated into a different flavour β at a given time t is $\langle \nu_\beta | \nu(t)\rangle$, or

$$P(\nu_\alpha \to \nu_\beta) = |\langle \nu_\beta | \nu(t)\rangle|^2 = \left| \sum_i U^*_{\alpha i} e^{-i\frac{m_i^2}{2E}L} U_{\beta i} \right|^2 \tag{10}$$

Eq (10) can be simplified to:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \mathfrak{Re}[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}] \sin^2\left(\frac{\Delta m^2_{ij} L}{4E}\right) + 2\sum_{i>j} \mathfrak{Im}[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}] \sin\left(\frac{\Delta m^2_{ij} L}{2E}\right) \tag{11}$$

where $\Delta m^2_{ij} = m_i^2 - m_j^2$ and $\mathfrak{Re}(z)$ and $\mathfrak{Im}(z)$ denote the real and imaginary parts of z, respectively. See the appendix for the complete derivation of this formula.

III.2. Analysis and Discussion

From Eq. (11), a few observations and remarks can be made about the nature of neutrino oscillations and more generally, neutrinos themselves.

1. The probability of oscillation depends on a sum of sine and sine-squared terms which are inherently oscillatory in nature. Therefore, the phenomenon is called neutrino *oscillation.*

2. The probability is 0 *only* if Δm^2_{ij} is zero; which is possible only if neutrinos are massless or all three mass states are identical. Since there is strong experimental evidence for the existence of oscillation, neutrinos clearly have non-zero mass and the three mass states are slightly different. Although exact neutrino masses have not been determined, we do know that there are two possible mass hierarchies: $m_1 < m_2 < m_3$, which is called the normal hierarchy or $m_3 < m_2 < m_1$ which is called the inverted mass hierarchy.

3. If there was no lepton mixing, all off-diagonal terms in the PMNS matrix would be zero, and at least one term in $U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}$ would be zero for $i > j$. Then Eq. (11) would reduce to $P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta}$. However, the existence of flavour change implies that lepton mixing occurs.

4. As the entire calculation was done in vacuum, the flavour change cannot arise from the interaction of

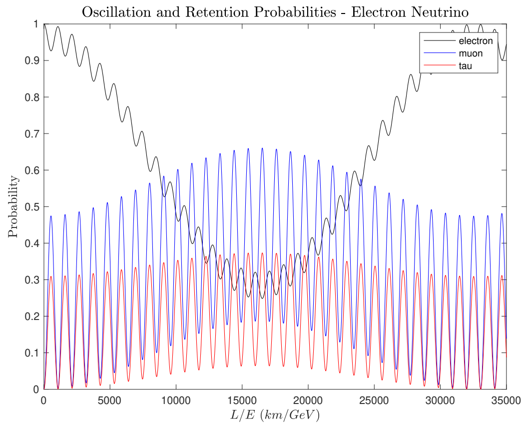

FIG. 1. A qualitative plot of the oscillation and retention probabilities for an electron neutrino to either change flavours into tau and muon neutrinos, or retain its flavour. The probabilities are plotted as a function of L/E (for long-range distances). Plot constructed using parameter values sourced from [10]

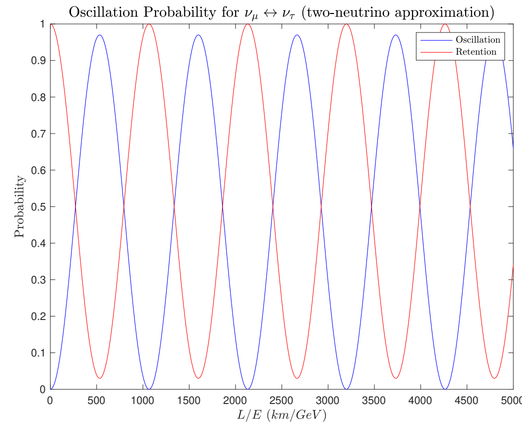

FIG. 2. A plot of the oscillation and retention probabilities for the process $\nu_\mu \leftrightarrow \nu_\tau$ as a function of L/E (for long-range distances) for the two-neutrino approximation. The probability oscillates as the neutrino travels in space. Plot constructed using parameter values sourced from [10]

neutrinos with matter. Therefore, oscillations are a completely inherent phenomenon that arise from the time evolution of the neutrino itself.

The survival probability is obtained by substituting α in place of β in Eq. (11). Therefore

$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4\sum_{i>j} |U_{\alpha i}U_{\alpha j}|^2 \sin^2\left(\frac{\Delta m_{ij}^2 L}{4E}\right) \quad (12)$$

because $U_{\alpha i}^* U_{\beta i} U_{\alpha j}^* U_{\alpha j}$ is real for $\alpha = \beta$.

III.3. The Two-Neutrino Case

While a detailed description of neutrino oscillations necessitates the use of the three-neutrino formula, for most practical purposes the two-neutrino case is sufficient. In this case, only two mass eigenstates ν_1 and ν_2 are significant. The PMNS matrix can be parameterised with just one mixing angle (two neutrino case does not require the CP violation phase [9]) θ. Therefore,

$$U = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad (13)$$

From Eq. (13) we see that $4U_{\alpha 2}^* U_{\beta 2} U_{\alpha 1} U_{\beta 1}^* = -4\sin\theta\cos\theta\cos\theta\sin\theta = -\sin^2 2\theta$. Eq. (11) is subsequently simplified to produce

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \quad (\alpha \neq \beta) \quad (14)$$

where $\Delta m^2 = m_{21}^2 = m_2^2 - m_1^2$.

IV. SYMMETRIES

Symmetries are physical quantities of a system that remain unchanged under a certain transformation. Some well-known examples of symmetries are the symmetry of a geometrical shape such as sphere, or the the fact that the speed of light remains unchanged in any reference frame. Symmetries are also intimately connected to conservation laws. From Noether's theorem, it can be said that every conservation law is associated with a symmetry of the system.
While translational and rotational invariance lead to symmetries of their own which form an important part of atomic and molecular physics, particle physics is primarily concerned with three important symmetries of the Hamiltonian — parity, charge conjugation, and time reversal.

IV.1. Parity Symmetry

First introduced by Wigner in 1927 [11], parity refers to the behaviour of a system under a spatial transformation

$$\mathbf{r} \to -\mathbf{r} \quad (15)$$

(in essence, a spatial reflection). This transformation is carried out by the parity operator $\hat{\pi}$. For a wave function $\langle r|\psi\rangle = \psi(\mathbf{r})$, then under the parity transformation,

$$\hat{\pi}\psi(\mathbf{r}) = \pi\psi(-\mathbf{r}) \quad (16)$$

The eigenvalues of the parity operator ($\pi = \pm 1$) are called the intrinsic parity of of the system.

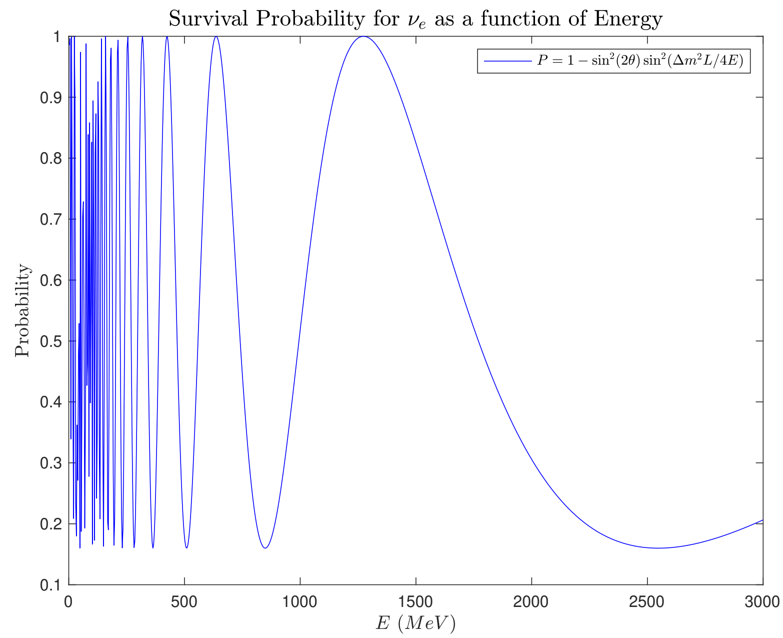

FIG. 3. A plot of the survival probability for an electron neutrino as a function of its energy (in MeV) for $L = 1800\,\text{km}$, $\Delta m^2 = 7.0 \times 10^{-5}\,\text{eV}^2$, and $\sin^2 2\theta = 0.84$. Plot constructed using parameter values sourced from [10]

IV.2. Charge Conjugation

Charge conjugation is the transformation where very particle is replaced by its antiparticle. Consider two classes of particles: a, particles that do not have distinct antiparticles, and b, particles that do. The action of the charge conjugation transformation by the operator $\hat{C}$ can be summarised as follows

$$\hat{C}\,|a,\psi_a\rangle = C_a\,|a,\psi_a\rangle \tag{17a}$$

$$\hat{C}\,|b,\psi_b\rangle = C_b\,|\bar{b},\psi_b^-\rangle \tag{17b}$$

Like the parity operator, the charge conjugation operator has eigenvalues $C = \pm 1$, called the C-parities.

IV.3. Time Reversal

Time-reversal symmetry is defined as invariance under the transformation

$$t \to -t \tag{18}$$

This symmetry is violated by weak interactions. Unlike charge conjugation and parity, this symmetry has no associated quantum number.

IV.4. CPT Symmetry

These three symmetries are fundamentally important in the field of particle physics. Although each of them has been broken in the present day universe, the Standard Model predicts that a combination of the three (that is the simultaneous application of all three transformations) must be a symmetry [12]. This combined symmetry is called the CPT symmetry. Since CPT as a whole is a symmetry, the breaking of any one must mean that the combination of other two is also broken.

One such violation is called the CP violation which is the violation of the combination of charge conjugation and parity. In the further section, we will cover the application of CP violation to the process of neutrino oscillations.

IV.5. CP Violations in Neutrino Oscialltions

Neutrinos, like all elementary particles, have a spin. As long as it isn't zero, the spin of a particle can either be *right-handed* or *left-handed*. This handedness can be determined by the direction in which your fingers curl if the thumb is placed parallel to the direction of propagation of the particle.

Neutrinos violate P-symmetry: Now, interestingly, all neutrinos we have observed so far are left-handed. No right-handed neutrinos have been found. Therefore, if you mirrored the entire universe, the laws of physics would change as there is no corresponding neutrino of opposite chirality. This constitutes a violation of P-symmetry.

Neutrinos violate C-symmetry: Now, let's look at a left-handed neutrino (since that's the only one we have ever observed). If you perform a parity transformation, you would expect to get a left-handed antineutrino. But there is one problem — left-handed antineutrinos have never been observed. So there is a violation of C-symmetry as well!

Therefore, it was important to look at CP symmetry as a whole. If you did that, a left-handed neutrino would produce a right-handed antineutrino under a CP transformation — something that does exist. This was assumed to patch up the asymmetry that was observed — until something more serious was found: evidence of CP violation itself. CP violation was found in phenomena such as Kaon decay or quark oscillations[2]. This violation was also incorporated into the quark mixing model.

CP violation is yet to be observed in neutrinos, but it is hypothesised that this asymmetry would manifest in neutrino oscillations. In fact, this has already been incorporated into the theory of the oscillations through the mixing model. As it was mentioned earlier, the PMNS matrix was parameterised using an angle denoted by δ_{CP} which was called the CP-violating phase. If δ_{CP} is not zero (or 180°), then CP violation exists in neutrino oscillations. The value of this phase signifies the extent of violation and affects how the neutrinos would oscillate between the three flavour states [13].

We will now discuss the probabilities of CP violation in the case of neutrino oscillations. Before we begin, let us summarise the effects of the C, P and T discrete symmetries on the oscillation probabilities.

$$P(\nu_\alpha \to \nu_\beta) \xrightarrow{CP} P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \tag{19a}$$

$$\xrightarrow{T} P(\nu_\beta \to \nu_\alpha) \tag{19b}$$

$$\xrightarrow{CPT} P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \tag{19c}$$

If we assume CPT conservations, that is

$$P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \tag{20}$$

(since the Standard Model predicts this), then the CP and T asymmetries in neutrino oscillations are equal. This can be quantified using a parameter $A_{\alpha\beta}$

$$\begin{aligned} A_{\alpha\beta} &= \frac{P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)}{P(\nu_\alpha \to \nu_\beta) + P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)} \\ &= \frac{P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha)}{P(\nu_\alpha \to \nu_\beta) + P(\nu_\beta \to \nu_\alpha)} \end{aligned} \tag{21}$$

Futhermore, there would be no CP violation in disappearance experiments, that is, $\nu_\alpha \to \nu_\alpha$ as $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$.

It is often convenient to introduce a quantity $\Delta P_{\alpha\beta}$, called the CP asymmetry, given by

$$\Delta P_{\alpha\beta} = P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \tag{22}$$

Performing some algebra on Eq. (11) and using Eq. (20), we find that this is nothing but twice the odd part of Eq. (11). We obtain that

$$\Delta P_{\alpha\beta} = 16J \sum_\gamma \varepsilon_{\alpha\beta\gamma} \sin\left(\frac{\Delta m_{21}^2 L}{4E}\right) \sin\left(\frac{\Delta m_{31}^2 L}{4E}\right) \sin\left(\frac{\Delta m_{32}^2 L}{4E}\right), \quad \text{where } J = \mathfrak{Im}[U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*] \tag{23}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol ($=1$ for an even permutation of (e,μ,τ)). The quantity J is called the *Jarlskog invariant* which is a measure of the CP violation. Using the standard parameterisation of the PMNS matrix, we obtain [14]

$$J = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\sin\delta_{CP} \tag{24}$$

This imposes some conditions on the parameters for the existence of CP violation in neutrino oscillations. It is required that the three mixing angles θ_{ij} be nonzero, and that the CP violating phase δ_{CP} lies between 0 and π. Furthermore, we also need the mass differences Δm_{ij}^2 to be non-vanishing. Consequently, all neutrino masses would have to be different. These conditions can be summarised below

$$\theta_{ij} \neq 0;\ \delta_{CP} \neq 0, \pi;\ m_1 \neq m_2, m_2 \neq m_3, m_1 \neq m_3 \tag{25}$$

Eq. (23) tells us some additional information too. As the effect of CP violation is proportional to $\sin\left(\frac{\Delta m_{21}^2 L}{4E}\right)$, this effect can only be observed in experiments that are sensitive to subdominant oscillations governed by Δm_{21}^2. Therefore, neutrino experiments that look for CP viola-

[2] For more details, see the Cabibo-Kobayashi-Maskawa matrix.

tions, like the DAEδALUS IsoDAR involve long distances (of the order of several hundred kilometres), intense neutrino beams and huge detectors.

V. CONCLUSION

In this work, we have looked at the phenomenon of neutrino oscillations, and studied the reasons for its occurrence. We have then derived the probability for a neutrino to oscillate between flavour states in both the two- and three- neutrino case. From this probability, we made a few observations and remarks about the neutrino itself. Next, we introduced the concept of discrete symmetries and looked at their relevance in quantum mechanics and particle physics. We also covered the concept of CPT invariance and CP violation. We connected CP violations to neutrino oscillations and quantified the CP asymmetry in terms of the Jarlskog invariant.

Appendix A: Derivation of $P(\nu_\alpha \to \nu_\beta)$

In this section, we cover the derivation of Eq. (11) at length. We know that the basis transformation relations between the flavour and mass eigenstates are given by the PMNS matrix. These have been summarised below for recapitulation.

$$
\begin{aligned}
|\nu_\alpha\rangle &= \sum_i U^*_{\alpha i} |\nu_i\rangle \\
|\nu_i\rangle &= \sum_\alpha U_{\alpha i} |\nu_\alpha\rangle
\end{aligned}
$$

We write Eq. (10) again,

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &= \left| \sum_i U^*_{\alpha i} U_{\beta i} \exp\left(\frac{-i m_i^2 L}{2E} \right) \right|^2 \\
&= \left[\sum_i U^*_{\alpha i} U_{\beta i} \exp\left(\frac{-i m_i^2 L}{2E} \right) \right] \left[\sum_j U^*_{\alpha j} U_{\beta j} \exp\left(\frac{-i m_j^2 L}{2E} \right) \right]^* \\
&= \sum_i \sum_j U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \exp\left(\frac{i \Delta m_{ji}^2 L}{2E} \right) \\
&= \sum_i U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i} + \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \exp\left(\frac{i \Delta m_{ji}^2 L}{2E} \right)
\end{aligned}
$$

We know that

$$
\begin{aligned}
e^{ix} &= \cos x + i \sin x \\
&= 1 - 2\sin^2 \frac{x}{2} + i \sin x
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &= \sum_i U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i} + \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \exp\left(\frac{i \Delta m_{ji}^2 L}{2E} \right) \\
&= \sum_i U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i} + \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} - 2 \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\frac{\Delta m_{ji}^2 L}{4E} \right) \\
&+ i \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\frac{\Delta m_{ji}^2 L}{2E} \right) \\
&= P_1 + P_2 - 2P_3 + iP_4
\end{aligned} \tag{A1}
$$

First, we consider the terms P_1 and P_2 from (A1).

$$\begin{aligned} P_1 + P_2 &= \sum_i U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i} + \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \\ &= \sum_{i,j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \\ &= \left(\sum_i U^*_{\alpha i} U_{\beta i} \right) \left(\sum_j U_{\alpha j} U^*_{\beta j} \right) \\ &= \left| \sum_i U_{\alpha i} U^*_{\beta j} \right|^2 \\ &= \delta_{\alpha\beta} \end{aligned} \tag{A2}$$

Where $\delta_{\alpha\beta}$ is the Kronecker delta function.

For P_3 and P_4 from (A1),

$$\begin{aligned} P_3 &= \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\frac{\Delta m^2_{ji} L}{4E} \right) \\ &= \sum_{i > j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\frac{\Delta m^2_{ij} L}{4E} \right) + \sum_{i < j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\frac{\Delta m^2_{ji} L}{4E} \right) \\ &= \sum_{i > j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\frac{\Delta m^2_{ij} L}{4E} \right) + \sum_{i > j} U^*_{\alpha j} U_{\beta j} U_{\alpha i} U^*_{\beta i} \sin^2\left(\frac{\Delta m^2_{ij} L}{4E} \right) \\ &= \sum_{i > j} \sin^2\left(\frac{\Delta m^2_{ij} L}{4E} \right) \left[(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) + (U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j})^* \right] \\ &= 2 \sum_{i > j} \mathfrak{Re}(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2\left(\frac{\Delta m^2_{ij} L}{4E} \right) \quad [z + z^* = 2\mathfrak{Re}(z)] \end{aligned} \tag{A3}$$

$$\begin{aligned} P_4 &= \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\frac{\Delta m^2_{ji} L}{2E} \right) \\ &= -\sum_{i > j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\frac{\Delta m^2_{ij} L}{2E} \right) + \sum_{i < j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\frac{\Delta m^2_{ji} L}{2E} \right) \\ &= -\sum_{i > j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\frac{\Delta m^2_{ij} L}{2E} \right) + \sum_{i > j} U^*_{\alpha j} U_{\beta j} U_{\alpha i} U^*_{\beta i} \sin\left(\frac{\Delta m^2_{ij} L}{2E} \right) \\ &= \sum_{i > j} \sin\left(\frac{\Delta m^2_{ij} L}{2E} \right) \left[(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j})^* - (U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \right] \\ &= -2i \sum_{i > j} \mathfrak{Im}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) \sin\left(\frac{\Delta m^2_{ij} L}{2E} \right) \quad [z^* - z = -2\mathfrak{Im}(z)] \end{aligned} \tag{A4}$$

Substituting (A2), (A3) and (A4) in (A1), we get:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i > j} \mathfrak{Re}[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}] \sin^2\left(\frac{\Delta m^2_{ij} L}{4E} \right) + 2 \sum_{i > j} \mathfrak{Im}[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}] \sin\left(\frac{\Delta m^2_{ij} L}{2E} \right) \tag{A5}$$

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