The Duffing-Oscillator Model: An Analysis of Inventory Fluctuations in Markets

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Abstract

The time series of financial fluctuations show complexity and chaos at different levels. In this paper we look at the inventory changes of wheat in the global market in the period between 1974 and 2012 and see how these fluctuations follow a nonlinear deterministic process. Using small sliding time windows, we find that the Duffing oscillator is able to model this process to a good degree of accuracy. This is mainly because there seems to be a cubic dependence of price changes on changes in stock. We also find that there is non-chaotic behaviour in this model.

1 Introduction

The study of chaotic motion in nonlinear systems has been a popular area of research during the last few decades. Many investigations have been performed on different nonlinear chaotic systems to understand the complex behavior of these systems. Three of the fundamental forced oscillators, Duffing, Van der Pol, and Rayleigh oscillators, have been extensively examined since lots of dynamic characteristics embedded in the physical systems can be realized from these three systems. Among them, forced Duffing oscillator is the most useful nonlinear dynamical systems, which is considered as a prototype model for various physical and engineering problems such as dynamics of a buckled elastic beam, particle in a forced double well, particle in a plasma, and a defect in solids.
Here we use the Duffing Oscillator Model to examine fluctuations of stocks of a particular commodity (here wheat). First we will analyze the various oscillator models and then eventually come up with the Duffing Oscillator Model. The reason for using this model has been thoroughly explained in the paper. The first section of this paper deals with mathematical modelling while the second part explains in detail the application of Duffing Oscillator in markets and other fields of economics in general. The third part is the analysis of data obtained from the original paper. All the analysis has been done using our knowledge of Nonlinear Dynamics as taught to us this semester.

2 Problem Definition and Mathematical Modelling

Nonlinear dynamic models have a long and venerable history in economic analysis, both theoretical and empirical. The objective here is to come up with a dynamical system which can closely mimic a fluctuation in supply and demand in a typical market.
In order to understand how a market works a step-by step approach has been taken so to better understand the mathematical model we are going to use in our study.
The first simple model will do some violence to the known strong inter-dependencies between markets and will not stress the role played by prices as equilibrating market forces. However, its very simplicity will enhance the ideas to be illustrated. Make note of the following symbols:

- X denotes the excess demand for a given commodity at a given point in time t
- $\dot{X}$ denotes the time derivative of X, it is the rate of excess demand
- S denotes the total stock size of the market(S is assumed to be fixed for the purposes of this analysis as it determines the scale within which the effect of excess demand)
- $\ddot{X}$ is the acceleration in the change in excess demand; $\ddot{X}$ is a continuously differentiable function of time t.

The basic idea here is that if economic forces are to react to a exogenous shock in excess demand in order to restore equilibrium, then to go to a zero velocity of excess demand to some non-zero velocity, the acceleration must first be non-negative to achieve this result.

2.1 Simple Harmonic Oscillator

Price(P) is an indicator function that translates demand and supply relationships into an excess demand function. If demand increases some economic force will act in the opposite direction to maintain equilibrium. The formulation is as follows:

$$\dot{P} = k\dot{X} \tag{1}$$

Therefore,

$$P = k_0 + kX \tag{2}$$

Where k and k_0 are constants to be determined by economic conditions.
If there has been displacement in equilibrium by an amount X then restoring force is assumed to be equal to $-\sigma X$, where σ is the 'coefficient of stiffness' in the restoring force. Mathematically this can be expressed as:

$$\ddot{X} = -\sigma X \tag{3}$$

or,

$$\ddot{X} + \sigma X = 0 \tag{4}$$

Note how this is similar simple harmonic motion. However, this model is very simple for practical use and cant be applied to mimic the market.

2.2 Forced Damped Oscillator

As our first step towards approaching reality, let us modify the equation.

$$\ddot{X} + \beta\dot{X} + \sigma X = 0 \tag{5}$$

Here $-\beta\dot{X}$ represents friction in the system.
Supose, now we consider the effect of a series of random shocks in the market system, say a sinusoidal path of shocks to the market system. Then we can define our system as:

$$\ddot{X} + \beta\dot{X} + \sigma X = \delta cos(\omega t) \tag{6}$$

Where ω is the frequency of the 'force equation' and δ is an appropriate constant.

2.3 Van der Pol Oscillator

Now let us move towards a nonlinear model.

- Let the retarding forced be modelled as: $-(\sigma_1 X + \sigma_3 X^3)X$
- Let the damping force be modelled as: $-\left[\beta_0\left(1 - \left(\frac{X}{X_0}\right)^2\right)\right]\dot{X}$

The resulting equation is the Van der Pol equation:

$$\ddot{X} + \left[\beta_0\left(1 - \left(\frac{X}{X_0}\right)^2\right)\right]\dot{X} + (\sigma_1 X + \sigma_3 X^3)X = 0 \tag{7}$$

Where $\beta_0 > 0$; $\sigma_1 > 0$; $\sigma_3 > 0$ or < 0
The economic interpretation is that for relatively small amounts of excess demand the restoring force is offset by the velocity term, but that if the excess demand gets to be too large, then the usual process applies

2.4 Duffing Oscillator

Now we are prepared to establish a model that can closely predict market behavior. Turns out that this model called 'The Duffing Oscillator model' is very well known in Physics. It basically describes the dynamics of a point mass in a double well potential. It is formulated as

$$\ddot{X} + \beta\dot{X} + \sigma X + \sigma_3 X^3 X = \delta cos(\omega t) \tag{8}$$

Or more generally,

$$\ddot{X} + \beta^{'}\dot{X} + \sigma_1{}^{'} X + \sigma_3{}^{'} X^3 X = \delta^{'} cos(\omega t) \tag{9}$$

Here, the cubic term is responsible for nonlinearity of the system.

3 Duffing Oscillator for Inventory Dynamics

Now that we have established our our model we are going to implement it. Dynamics of supply and demand tends to be complex in explaining the observed inventory volatility. External forces (like policy) and other disturbances play a major role in the dynamics of supply and demand

The commodity production cycle consists of two negative feedback loops, consumption and production. Say, if the inventories of a commodity fall the price of the commodity increases and vice-versa. Thus the price-stock relation is a push-pull effect in which price change acts as a restoring force driven by oscillations of the inventory about the reference level. In simple words the market responds in a way such that equilibrium is retained. The rate of change in price change of a commodity can be modelled in a nonlinear way.

$$\dot{p} = \alpha_1 x + \alpha_2 x^3 \qquad \alpha_1 \alpha_2 < 0 \tag{10}$$

Here $\dot{p}$ represents the price change per year. The dynamic of production tells us that $x \approx P(p,x) - C(p,x)$, also $\dot{x} \propto -x$, therefore

$$\dot{x} = r(P(p,x) - C(p,x)) \tag{11}$$

P, C represent the production and consumption functions respectively and $r \leq 0$ is the constant of proportionality. Eqs.(10), (11) represent the coupled feed-back loops of commodity of oscillations.

The external force which has a certain frequency is responsible for the eventual stabilization of fluctuations caused in the market. The force may be a policy intervention in general, but there are other forces at play too. Let us suppose that the force is of the form $a\sin(\omega t)$.

At time $t_0 = n\pi$ the force is zero; $n = 0, 1, 2, 3.....$
Superposition in Eq.(11) leads to the following

$$\dot{x} = rP(p(t), x(t)) - C(p(t), x(t)) + a\sin(\omega t) \tag{12}$$

Taking the time derivative of Eq.(12) gives

$$\ddot{x} = r\left(\frac{\partial P}{\partial p}\dot{p} + \frac{\partial P}{\partial x}\dot{x} - \frac{\partial C}{\partial p}\dot{p} - \frac{\partial C}{\partial x}\dot{x} + a\omega\cos(\omega t)\right) \tag{13}$$

Let:

- $\delta = -r\frac{\partial(P-C)}{\partial x}$
- $\beta = -r\alpha_1\frac{\partial(P-C)}{\partial x}$
- $\alpha = -r\alpha_2\frac{\partial(P-C)}{\partial x}$
- $\gamma = ra\omega$

Now substituting in Eq.(13) we get,

$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma cos(\omega t) \tag{14}$$

Clearly Eq.(14) is the Duffing oscillator equation, an example of damped physical oscillations which may or may not be chaotic. This is one of the main questions to be answered. Is the behaviour of the system chaotic or not? It all depends on the parameters δ, β, α and λ. Here δ represents extent of economic damping; β, γ represent nonlinear price-stock push and pull respectively; γ represents the amplitude of the external force.

Now we are in a position to analyze the behaviour of inventory dynamics. Let $\dot{x} = y, t = z$ and $\ddot{x} = y$, then we can write Eq.(14) as a 3D system

$$\dot{x} = y \tag{15}$$

$$\dot{y} = -\delta y - \beta x - \alpha x^3 + \gamma cos(\omega t)$$

$$\dot{z} = 1$$

Note that δ represents economic damping, $\beta > 0$ is the measure of push towards equilibrium, whereas $\alpha < 0$ is the pull factor, and γ is the amplitude of force.

3.1 Analysis

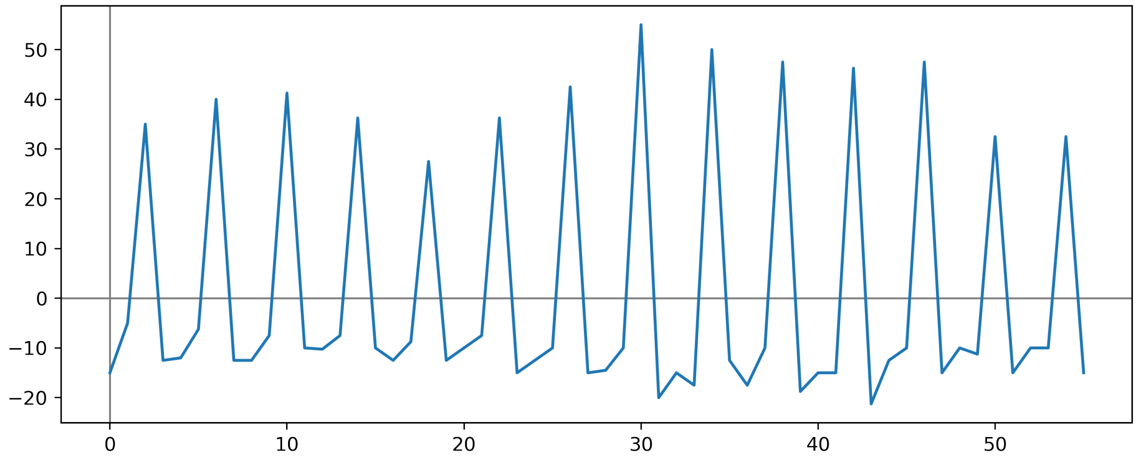
In a typical Duffing Equation potential is given by $V = \frac{-x^2}{2} + \frac{x^4}{4}$ and $\frac{\partial V}{\partial x} = -x + x^3$. Here the partial derivative of potential is given by change in price as in Eq.(10)

$$\dot{p} = \alpha_1 x + \alpha_2 x^3 \qquad \alpha_1 \alpha_2 < 0$$

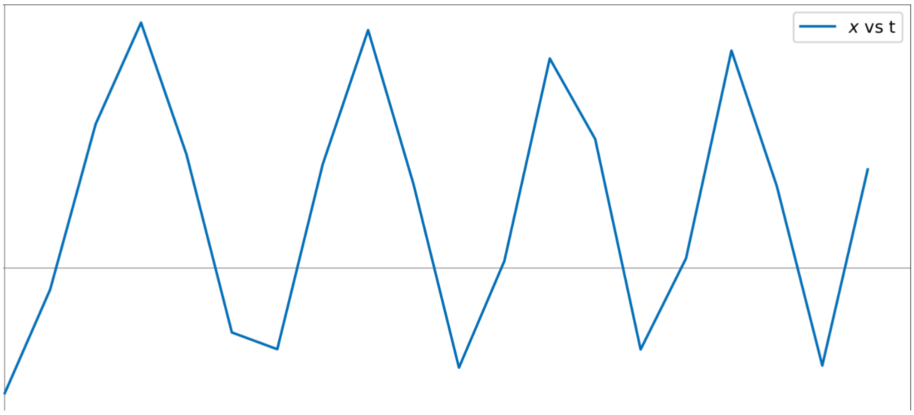
$\alpha_1 < 0, \alpha_2 > 0$. Which subsequently makes $\beta > 0$ and $\alpha < 0$.

The economic damping term is defined as $\delta = -r\frac{\partial(P-C)}{\partial x}$. $\delta < 0$ signifies that amplitude oscillations increase with time. This corresponds to a situation when a rise in inventory is accompanied by a reduction in the gap between production and consumption due to speculation about price rise. On the other hand if $\delta > 0$ the oscillations decrease in amplitude with time. In a duffing oscillator model $\delta < 0$ is a signature of chaos.

The following is the data obtained for quarterly fluctuations of stocks of wheat during the period (1974-1999) which correspond to 55 points on the x axis (0 denoting 1974 and 55 denoting 1999).



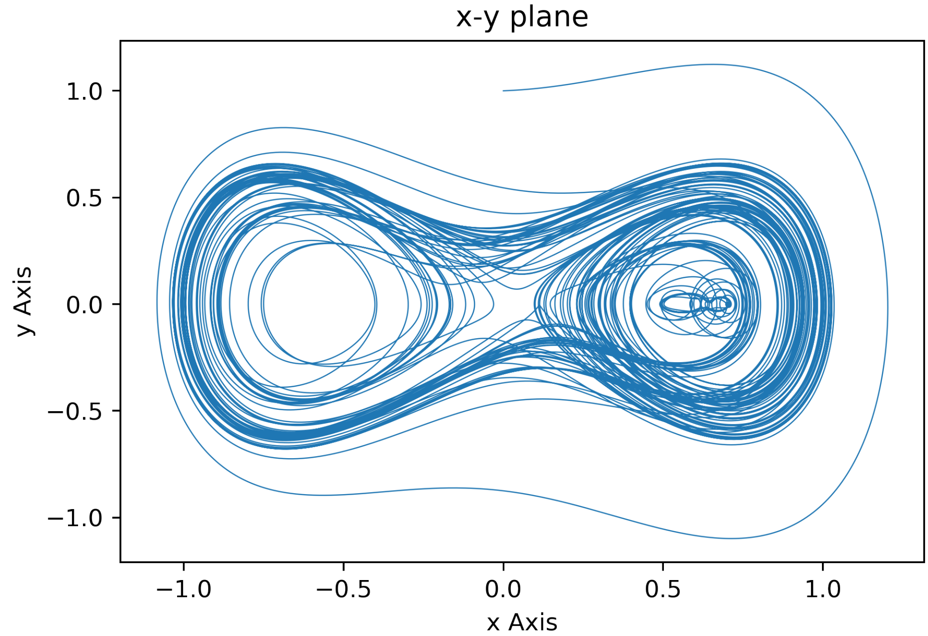
Seeing the behavior of the graph we can presume that the damping term is close to zero other wise the oscillations tend to decrease after a while which isn't the case. Therefore, for our model we are going to assume δ as very close to zero. However, this method can only be applied to model local behaviour and not global behaviour of change in stocks. The author of the original paper used sliding time windows calculate the parameters using the time-series estimation method. As this is beyond the scope of this course we shall tackle this problem differently.

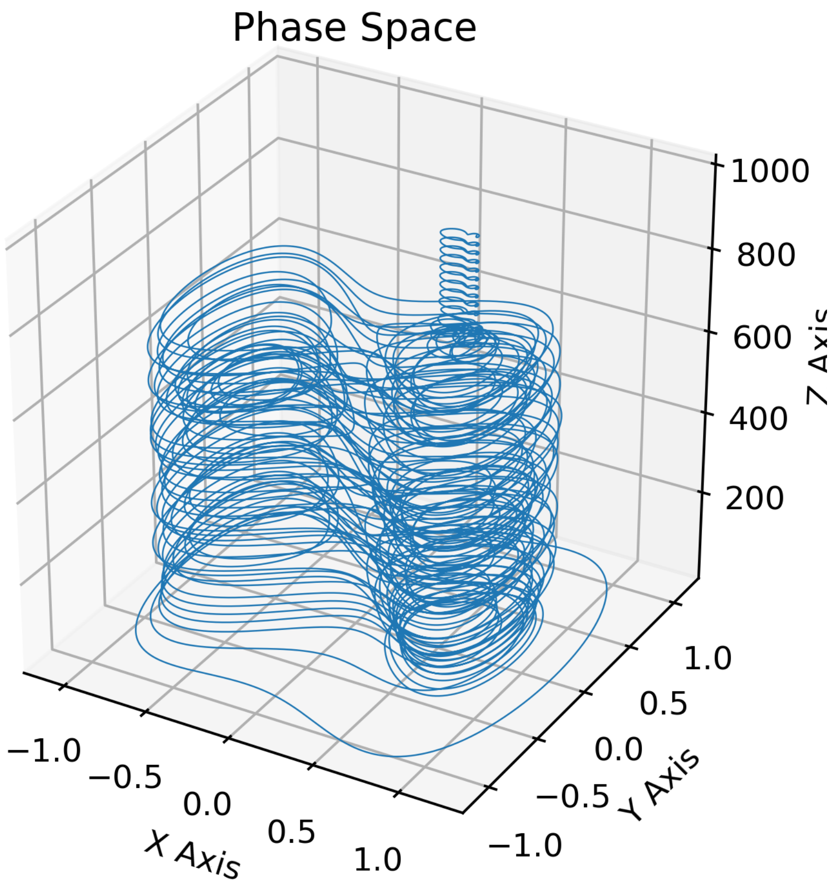


This graph was obtained for $\delta = 0.01$ for the first five years of data. The other parameters were set according to a typical duffing oscillator model. Note that this graph is just presented to show that the behavior of our proposed duffing oscillator model is the same as the set of data observed. To obtain a full graph we need to obtain different parameter values for different time windows. One parameter however remains almost the same, $\delta \approx 0$.

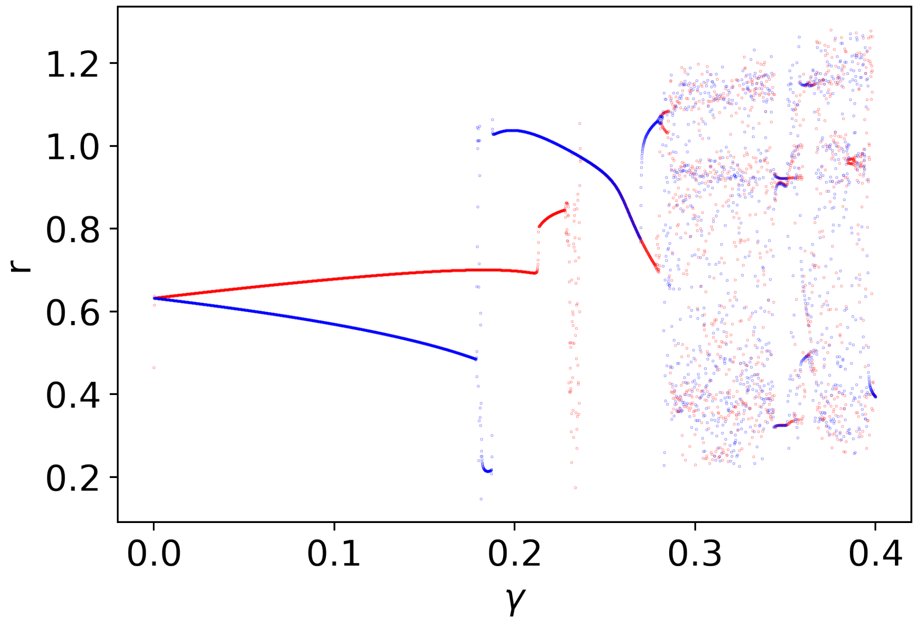
3.2 Results

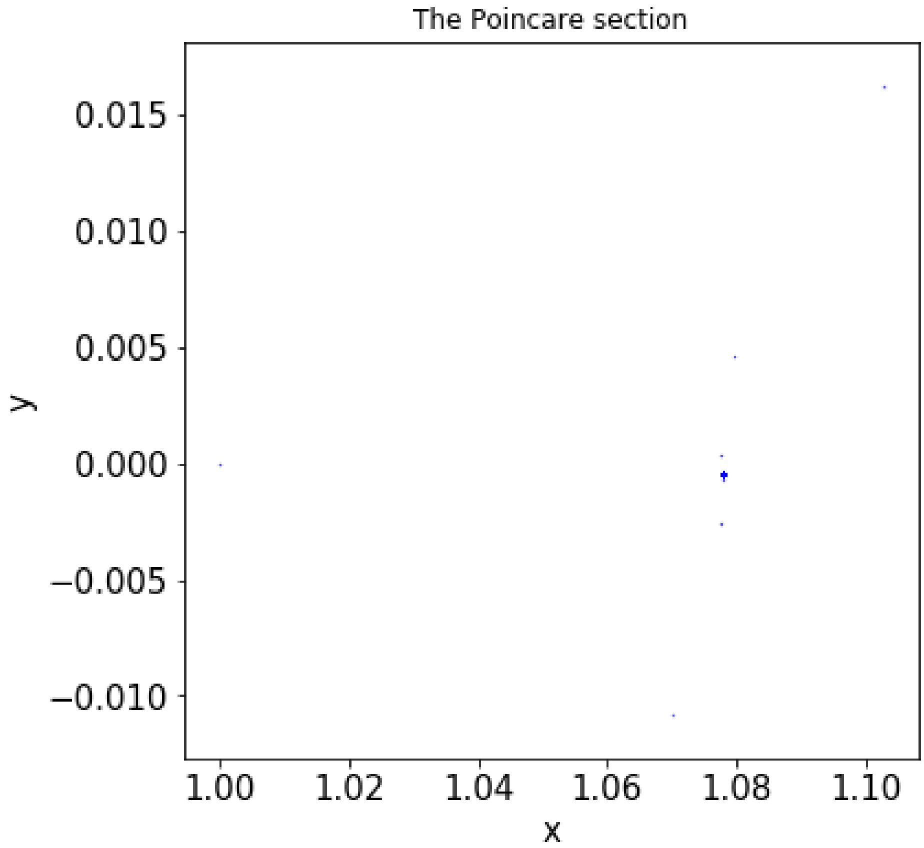
The following graphs were obtained for $\delta = 0.1, \alpha = 2.5, \beta = -1, \gamma = 0.17, \omega = 0.314$.





The following is the bifurcation diagram.





$\gamma = 0.17$ does not lie in the chaotic range (Bifurcation Diagram) so we can safely say that behavior is not chaotic.

3.3 Limitations of the approach

The approach taken to model oscillations of change of stocks with respect to time has some drawbacks. The method used here can only approximate the behaviour of oscillations for small time periods as the value of parameters differs for all sliding time windows. More accurate result is obtained if one keeps shrinking the size of time windows. More accurate frequency is obtained for smaller time windows. The author of the paper has used advanced numerical computations such as Fourier time series estimation to model data which is beyond the scope of the course. Therefore a different approach was taken to produce the same result.

4 Discussion and Conclusions

The oscillatory behavior of quarterly fluctuations of wheat inventories in the global market exhibits a complex non-random character. The contribution of the present study lies in establishing that the deterministic Duffing Oscillator is able to explain the dynamics of inventory fluctuations of wheat for a given time period, with reasonable credibility. This stems mainly from the evidence of a cubic dependence of changes in price on those of stocks. Also, it facilitates a better prediction by economic agents.

As is observed for $\delta \approx 0$ non chaotic behavior is observed. For $\delta < 0$ the situation is such that a rise in inventory is accompanied by a reduction in the gap between production and consumption due to speculation about price rise. But the graph obtained tells us that delta values remain very close to zero throughout the whole time scale otherwise a large negative delta value would lead to the oscillations blowing up in time, on the other hand negative delta values will result in a decay in oscillations. None of these two is observed therefore it can be safely said that delta values are always nearly zero for different time windows signifying non-chaotic behaviour.

Appendices

All the graphs and numerical data have been computed using Python in Jupyter Notebook. A copy of the Jupyter Notebook has also been attatched along with this report for reference.

Code:

Behavior of change of stocks vs t, Phase space and plane

```
x vs t (Behavior of change of stocks)
from scipy.integrate import odeint

import datetime
import numpy as np
import scipy as sp
import scipy.fftpack
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline



alpha = 2.5
beta = -1
gamma = 0.17
delta = 0.1
omega = 0.314

def model(w,t):
    dxdt = w[1]
    dydt = -delta*w[1] - beta*w[0] - alpha*w[0]**3 + gamma*np.cos(omega*t)
    dzdt = 1
    dwdt = [dxdt, dydt, dzdt]
    return dwdt

#initial conditions
w0 = [-15,0,0]

#time points
xlim=5
t = np.linspace(0,xlim,20)

#solve ODE
```

```
w = odeint(model,w0,t)


# original graph
data = pd.read_csv('/home/adhmgn/snap/libreoffice/177/stocks.csv')
df = pd.DataFrame(data, columns= ['year','ch_stocks'])
time = data.year
stocks = data.ch_stocks




#plotting
fig = plt.figure(figsize=(10,15), dpi=260)

plt.subplot(3, 1, 1)
plt.axhline(y=-5,lw=0.5, color='gray')
plt.axvline(lw=0.5, color='gray')
plt.plot(t,w[:,0], label = '$x$ vs t')
#plt.plot(time, stocks)
#plt.xlim(50,100)
plt.xlabel('time')
plt.ylabel('$x$')
plt.legend(loc='best')




plt.show()

from mpl_toolkits.mplot3d import Axes3D


def lorenz(x, y, z):
    '''
    Given:
       x, y, z: a point of interest in three dimensional space
       s, r, b: parameters defining the lorenz attractor
    Returns:
       x_dot, y_dot, z_dot: values of the lorenz attractor's partial
```

```
            derivatives at the point x, y, z
    '''
    x_dot = y
    y_dot = -delta*y - beta*x - alpha*x**3 + gamma*np.cos(omega*z)
    z_dot = 1
    return x_dot, y_dot, z_dot


dt = 0.01
num_steps = 100000

# Need one more for the initial values
xs = np.empty(num_steps + 1)
ys = np.empty(num_steps + 1)
zs = np.empty(num_steps + 1)

# Set initial values
xs[0], ys[0], zs[0] = (0., 1., 1.05)

# Step through "time", calculating the partial derivatives at the current point
# and using them to estimate the next point
for i in range(num_steps):
    x_dot, y_dot, z_dot = lorenz(xs[i], ys[i], zs[i])
    xs[i + 1] = xs[i] + (x_dot * dt)
    ys[i + 1] = ys[i] + (y_dot * dt)
    zs[i + 1] = zs[i] + (z_dot * dt)


# Plot
fig,ax = plt.subplots(1,1, dpi=360)
ax.plot(xs, ys, lw=0.5)
ax.set_xlabel("x Axis")
ax.set_ylabel("y Axis")
ax.set_title("x-y plane")


fig = plt.figure(figsize=(5,5), dpi=360)
ax = fig.gca(projection='3d')
ax.plot(xs, ys, zs, lw=0.5)
ax.set_xlabel("X Axis")
ax.set_ylabel("Y Axis")
```

```
ax.set_zlabel("Z Axis")
ax.set_title("Phase Space")

plt.show()
```

Bifurcation and Poincare Map

```
# Program 09c: Phase portrait and Poincare section of a nonautonomous ODE.
# See Figure 9.11(b).

import matplotlib.pyplot as plt
import numpy as np
from scipy.integrate import odeint
xmin, xmax = -2, 2
ymin, ymax = -2, 2

gamma = 0.17

def dx_dt(x, t):
    return [x[1], x[0] - k*x[1] - x[0]**3 + gamma*np.cos(omega*t)]


# The Poincare section.
fig, ax = plt.subplots(figsize=(6, 6))
t = np.linspace(0, 4000 * (2*np.pi) / omega, 16000000)
xs = odeint(dx_dt, [1, 0], t)

x = [xs[4000*i, 0] for i in range(4000)]
y = [xs[4000*i, 1] for i in range(4000)]

ax.scatter(x, y, color='blue', s=0.1)
plt.xlabel('x', fontsize=15)
plt.ylabel('y', fontsize=15)
plt.tick_params(labelsize=15)
plt.title('The Poincare section')
plt.show()
```